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The entangled ergodic theorem in the almost periodic case

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A R T I C L E I N F O

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A B S T R A C T

Let U be a unitary operator acting on the Hilbert space $\mathcal{H}$, and $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$ a pair partition. Then the ergodic average

$$\frac{1}{N^k}\sum_{n_1,\ldots,n_k=0}^{N-1} U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(2k-1)}}A_{2k-1}U^{n_{\alpha(2k)}}$$

converges in the strong operator topology provided U is almost periodic, that is when $\mathcal{H}$ is generated by the eigenvalues of U. We apply the present result to obtain the convergence of the Cesaro mean of several multiple correlations.

1. Introduction

An entangled ergodic theorem was introduced in [1] in connection with the quantum central limit theorem, and clearly formulated in [6]. Namely, let U be a unitary operator on the Hilbert space $\mathcal{H}$, and for $m \geqslant k$, $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ a partition of the set $\{1, \ldots, m\}$ in k parts. The *entangled ergodic theorem* concerns the convergence in the strong, or merely weak (s-limit, or w-limit for short) operator topology, of the multiple Cesaro mean

$$\frac{1}{N^k}\sum_{n_1,\ldots,n_k=0}^{N-1} U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(m-1)}}A_{m-1}U^{n_{\alpha(m)}}, \tag{1.1}$$

$A_1, \ldots, A_{m-1}$ being bounded operators acting on $\mathcal{H}$.

Expressions like (1.1) naturally appear in the study of multiple correlations, see Section 4. The simplest case is nothing but the well known mean ergodic theorem due to John von Neumann

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$$\text{s-}\lim_N \frac{1}{N}\sum_{n=0}^{N-1} U^n = E_1, \tag{1.2}$$

E_1 being the selfadjoint projection onto the eigenspace of the invariant vectors for U. The entangled ergodic theorem in not yet available, and it is expected to fail in the full generality (see, e.g. [7, p. 8]). In addition, it is yet unknown what are general enough conditions under which it can be proved. In [2] it is shown that the entangled ergodic theorem holds true in the case when the A_j in (1.1) are compact, without any condition on the unitary U, and in the almost periodic case (i.e. when $\mathcal{H}$ is generated by the eigenvalues of U) for some very special pair partitions, without any condition on the A_j. Another interesting case arising from "quantum diagonal measures" is treated in [3].

In the present note we prove that the entangled ergodic theorem holds true in the almost periodic case. Namely, the Cesaro mean in (1.1) converges in the strong operator topology for all the pair partitions α, provided the dynamics generated by the unitary U on the Hilbert space $\mathcal{H}$ is almost periodic. We apply the present result to obtain the convergence of the Cesaro mean of several multiple correlations for C^*-dynamical systems such that the unitary implementing the dynamics in the GNS Hilbert space is almost periodic.

For the sake of completeness, we report the analogous result involving the multiple correlations for C^*-dynamical systems based on compact operators.

2. Notations and basic facts

Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator acting on the Hilbert space $\mathcal{H}$. The unitary U is said to be *almost periodic* if $\mathcal{H} = \mathcal{H}^U_{\text{ap}}$, $\mathcal{H}^U_{\text{ap}}$ being the closed subspace consisting of the vectors having relatively norm-compact orbit under U. It is seen in [7] that U is almost periodic if and only if $\mathcal{H}$ is generated by the eigenvectors of U. Denote $\sigma(U)$ and $\sigma_{\text{pp}}(U) \subset \sigma(U)$ the spectrum and the pure point spectrum (i.e. set of all the eigenvalues of U) of U respectively. Define

$$\sigma^{\text{a}}_{\text{pp}}(U) := \left\{ z \in \sigma_{\text{pp}}(U) \middle| zw = 1 \text{ for some } w \in \sigma_{\text{pp}}(U)\right\},$$

that is the "antidiagonal" part of $\sigma_{\text{pp}}(U)$. A partition $\alpha : \{1,\dots,m\} \mapsto \{1,\dots,k\}$ of the set made of m elements in k parts is nothing but a surjective map, the parts of $\{1,\dots,m\}$ being the preimages $\{\alpha^{-1}(\{j\})\}_{j=1}^k$. A *pair partition* is nothing but a partition such that the preimages are made by two elements.

Consider, for each finite subset $F \subset \sigma^{\text{a}}_{\text{pp}}(U)$ and $\{A_1,\dots,A_{2k-1}\} \subset \mathcal{B}(\mathcal{H})$, the following operator

$$S^F_{\alpha;A_1,\dots,A_{2k-1}} := \sum_{z_1,\dots,z_k \in F} E_{z^\#_{\alpha(1)}} A_1 E_{z^\#_{\alpha(2)}} \cdots E_{z^\#_{\alpha(2k-1)}} A_{2k-1} E_{z^\#_{\alpha(2k)}} \tag{2.1}$$

together with the sesquilinear form

$$s^F_{\alpha;A_1,\dots,A_{2k-1}}(x,y) := \left\langle S^F_{\alpha;A_1,\dots,A_{2k-1}} x, y \right\rangle,$$

where the pairs $z^\#_{\alpha(i)}$ are alternatively z_j and $\bar{z}_j$ whenever $\alpha(i) = j$, and E_z is the selfadjoint projection on the eigenspace corresponding to the eigenvalue $z \in \sigma_{\text{pp}}(U)$. If for example, α is the pair partition $\{1,2,1,2\}$ of four elements, we write (cf. Proposition 2.3) for the limit in the strong operator topology of (2.1),

$$S_{\alpha;A,B,C} = \sum_{z,w \in \sigma^{\text{a}}_{\text{pp}}(U)} E_{\bar{z}} A E_{\bar{w}} B E_z C E_w.$$

The strong limits of the operators $S^F_{\alpha;A_1,\dots,A_{2k-1}}$ will describe the limit of the Cesaro means (1.1) in the case under consideration in the present paper, see Theorem 3.1. The reader is referred to [3] for the case when the A_j are compact operators.

We report the proof of the following results for the convenience of the reader.

Lemma 2.1. *We have for the above sesquilinear form,*

$$\left|s^F_{\alpha;A_1,\ldots,A_{2k-1}}(x,y)\right| \leqslant \|x\|\,\|y\| \prod_{j=1}^{2k-1} \|A_j\|,$$

uniformly for F finite subsets of $\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)$.

Proof. The proof follows by the repeated application of

$$\left\|\sum_{j\in J} P_j\xi_j\right\|^2 = \sum_{j\in J} \|P_j\xi_j\|^2,$$

by taking into account the Schwarz and Bessel inequalities. Here, $\{P_j\}_{j\in J}$ is any orthogonal set of selfadjoint projections acting on a Hilbert space $\mathcal{H}$, and $\{\xi_j\}_{j\in J} \subset \mathcal{H}$. The reader is referred to [2] to see how the proof works in a pivotal case. $\square$

Lemma 2.2. *The net* $\left\{\sum_{z\in F} E_{\bar{z}}AE_z \,|\, F \text{ finite subset of } \sigma^{\mathrm{a}}_{\mathrm{pp}}(U)\right\}$ *converges in the strong operator topology.*

Proof.

$$\left\|\sum_{z\in F} E_{\bar{z}}AE_z x - \sum_{z\in G} E_{\bar{z}}AE_z x\right\| \leqslant \left\|\sum_{z\in F\backslash G} E_{\bar{z}}AE_z x\right\| + \left\|\sum_{z\in G\backslash F} E_{\bar{z}}AE_z x\right\|.$$

By taking into account Lemma 2.1, it is enough to prove that for $\varepsilon > 0$, there exists a finite set G_ε, such that $\|\sum_{z\in H} E_{\bar{z}}AE_z x\| < \frac{\varepsilon}{3}$ whenever $H \subset G^c_\varepsilon$. But,

$$\left\|\sum_{z\in H} E_{\bar{z}}AE_z x\right\|^2 = \sum_{z\in H} \|E_{\bar{z}}AE_z x\|^2 \leqslant \|A\|^2 \sum_{z\in H} \|E_z x\|^2.$$

The proof follows as the last sum is convergent. $\square$

Proposition 2.3. *For each finite set* $\{A_1,\ldots,A_{2k-1}\} \subset \mathcal{B}(\mathcal{H})$, *the net* $\{S^F_{\alpha;A_1,\ldots,A_{2k-1}} | F$ *finite subset of* $\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)\}$ *converges in the strong operator topology to a element in* $\mathcal{B}(\mathcal{H})$ *denoted by* $S_{\alpha;A_1,\ldots,A_{2k-1}}$.

Proof. As for any finite $F \subset \sigma^{\mathrm{a}}_{\mathrm{pp}}(U)$,

$$S^F_{\alpha;A_1,\ldots,A_{2k-1}} = ES^F_{\alpha;A_1,\ldots,A_{2k-1}}E,$$

E being the selfadjoint projection onto the almost periodic subspace of U, we suppose without loss of generality, that $x \in \mathcal{H}$ is an eigenvector of U with eigenvalue z_0. The proof is by induction on k. By Lemma 2.2, it is enough to show that the assertion holds true for the pair partition $\beta : \{1,\ldots,2k+2\} \mapsto \{1,\ldots,k+1\}$, provided it is true for any pair partition $\alpha : \{1,\ldots,2k\} \mapsto \{1,\ldots,k\}$. Let $k_\beta \in \{1,\ldots,2k+2\}$ be the first element of the pair $\beta^{-1}(\{k+1\})$, and α_β the pair partition of $\{1,\ldots,2k\}$ obtained by deleting $\beta^{-1}(\{k+1\})$ from $\{1,\ldots,2k+2\}$, and $k+1$ from $\{1,\ldots,k+1\}$. We obtain

$$S^F_{\beta;A_1,\ldots,A_{2k+1}}x = S^F_{\alpha_\beta;A_1,\ldots,A_{k_\beta-1}E_{\bar{z}_0}A_{k_\beta},\ldots,A_{2k}}A_{2k+1}x,$$

provided $\bar{z}_0 \in F$. We get

$$\lim_{F\uparrow\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)} S^F_{\beta;A_1,\ldots,A_{2k+1}}x = S_{\alpha_\beta;A_1,\ldots,A_{k_\beta-1}E_{\bar{z}_0}A_{k_\beta},\ldots,A_{2k}}A_{2k+1}x,$$

$S_{\alpha;A_1,\ldots,A_{2k-1}}$ being the limit in the strong operator topology of $S^F_{\alpha;A_1,\ldots,A_{2k-1}}$ which exists by the inductive hypothesis. $\square$

We symbolically write

$$\begin{aligned}S_{\alpha;A_1,\ldots,A_{2k-1}} &:= \underset{F\uparrow\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)}{\text{s-lim}}\, S^F_{\alpha;A_1,\ldots,A_{2k-1}}\\ &= \sum\nolimits_{z_1,\ldots,z_k\in\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)} E_{z^{\#}_{\alpha(1)}}A_1E_{z^{\#}_{\alpha(2)}}\cdots E_{z^{\#}_{\alpha(2k-1)}}A_{2k-1}E_{z^{\#}_{\alpha(2k)}},\end{aligned} \tag{2.2}$$

where in (2.2) the pairs $z^{\#}_{\alpha(i)}$ are alternatively z_j and $\bar z_j$ whenever $\alpha(i)=j$ as in (2.1). By Lemma 2.1, we get

$$\|S_{\alpha;A_1,\ldots,A_{2k-1}}\| \leqslant \prod_{j=1}^{2k-1}\|A_j\|. \tag{2.3}$$

3. The entangled ergodic theorem in the almost periodic case

In the present section we prove the entangled ergodic theorem for the almost periodic situation. In this way, we improve the results in Section 3 of [2] where only very special pair partitions were considered. We suppose that $\mathcal{H}$ is generated by the eigenvectors of U if it is not otherwise specified.

The proof of the following result relies upon the mean ergodic theorem (1.2), by showing that, step by step, one can reduce the matter to the dense subspace algebraically generated by the eigenvectors of U.

We start by pointing out some preliminary facts on the pair partition $\alpha:\{1,2.\ldots,2k\}\mapsto\{1,2.\ldots,k\}$ used in the proof. We can put

$$\{1,2,\ldots,2k\}=\{i_1,i_2,\ldots,i_k\}\bigcup\{j_k,\ldots,j_2,j_1\}$$

with $j_k<j_{k-1}<\cdots<j_2<j_1=2k$, $\alpha^{-1}(\{1\})=\{i_1,2k\}$, the order of the set $\{i_1,i_2.\ldots,i_k\}$ is that determined by α, i_m is the greatest element of $\{i_1,i_2,\ldots,i_k\}$ (perhaps possibly coinciding with i_1), and finally j_h is the first element after i_m (i.e. $i_m+1=j_h$). Namely, "$\bigcup$" stands for disjoint union, and $\alpha^{-1}(\{l\})=\{i_l,j_l\}$ with $i_l<j_l$, $l=1,2,\ldots,k$.

Theorem 3.1. *Let U be an almost periodic unitary operator acting on the Hilbert space $\mathcal{H}$. Then for each pair partition $\alpha:\{1,\ldots,2k\}\mapsto\{1,\ldots,k\}$, and $A_1,\ldots,A_{2k-1}\in\mathcal{B}(\mathcal{H})$,*

$$\begin{aligned}&\underset{N\to+\infty}{\text{s-lim}}\left\{\frac{1}{N^k}\sum_{n_1,\ldots,n_k=0}^{N-1}U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(2k-1)}}A_{2k-1}U^{n_{\alpha(2k)}}\right\}\\ &\quad=S_{\alpha;A_1,\ldots,A_{2k-1}}.\end{aligned}$$

Proof. We suppose without loss of generality (cf. (2.3)), that $\|A_i\|\leqslant 1$, $i=1,\ldots,2k-1$. Fix $\varepsilon>0$, and choose recursively the following sets. Let I_ε be such that

$$\left\|x-\sum_{\eta_1\in I_\varepsilon}E_{\eta_1}x\right\|<\varepsilon.$$

For each $\eta_1\in I_\varepsilon$, let $I_\varepsilon(\eta_1)$ be such that

$$\left\|A_{2k-1}E_{\eta_1}x-\sum_{\eta_2\in I_\varepsilon(\eta_1)}E_{\eta_2}A_{2k-1}E_{\eta_1}x\right\|<\frac{\varepsilon}{|I_\varepsilon|},$$

provided $i_1 < j_2$.[1] Finally, for each $\eta_1 \in I_\varepsilon, \eta_2 \in I_\varepsilon(\eta_1), \ldots, \eta_{k-1} \in I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2})$, let $I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2}, \eta_{k-1})$ be such that

$$\Big\| A_{j_k+1} E_{\eta^\#_{\alpha(j_k+1)}} \cdots E_{\eta_2} A_{2k-1} E_{\eta_1} x - \sum_{\eta_k \in I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2}, \eta_{k-1})} E_{\eta_k} A_{j_k} A_{j_k+1} E_{\eta^\#_{\alpha(j_k+1)}} \cdots E_{\eta_2} A_{2k-1} E_{\eta_1} x \Big\| < \frac{\varepsilon}{\sum_{\eta_1 \in I_\varepsilon} \sum_{\eta_2 \in I_\varepsilon(\eta_1)} \cdots \sum_{\eta_{k-1} \in I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2})} |I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2}, \eta_{k-1})|}.$$

By taking into account (1.2), choose N_ε such that

$$\left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} (\eta_m U)^n - E_{\bar{\eta}_m} \right) A_{i_m} E_{\eta_h} \cdots E_{\eta_2} A_{2k-1} E_{\eta_1} x \right\| < \frac{\varepsilon}{\sum_{\eta_1 \in I_\varepsilon} \sum_{\eta_2 \in I_\varepsilon(\eta_1)} \cdots \sum_{\eta_{h-1} \in I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{h-2})} |I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{h-2}, \eta_{h-1})|},$$

and after $k-1$ steps,

$$\left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} (\eta_k U)^n - E_{\bar{\eta}_k} \right) A_1 E_{\eta^\#_{\alpha(2)}} \cdots E_{\eta_2} A_{2k-1} E_{\eta_1} x \right\| < \frac{\varepsilon}{\sum_{\eta_1 \in I_\varepsilon} \sum_{\eta_2 \in I_\varepsilon(\eta_1)} \cdots \sum_{\eta_{k-1} \in I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2})} |I_\varepsilon(\eta_1, \eta_2, \ldots, \eta_{k-2}, \eta_{k-1})|},$$

whenever $N > N_\varepsilon$. We then have

$$\begin{aligned}
&\left\| \left(\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(2k-1)}} A_{2k-1} U^{n_{\alpha(2k)}} - S_{\alpha; A_1, \ldots, A_{2k-1}} \right) x \right\| \\
&\leqslant \left\| \left(\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(2k-1)}} A_{2k-1} U^{n_{\alpha(2k)}} - S_{\alpha; A_1, \ldots, A_{2k-1}} \right) \right\| \\
&\quad \times \left\| \left(x - \sum_{\eta_1 \in I_\varepsilon} E_{\eta_1} x \right) \right\| \\
&\quad + \Big\| \sum_{\eta_1 \in I_\varepsilon} \Big(\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots A_{i_1-1} (\eta_1 U)^{n_1} A_{i_1} \cdots U^{n_{\alpha(2k-1)}} \\
&\quad - \sum_{z_2, \ldots, z_k \in \sigma^{\mathrm{a}}_{\mathrm{pp}}(U)} E_{z^\#_{\alpha(1)}} A_1 E_{z^\#_{\alpha(2)}} \cdots A_{i_1-1} E_{\bar{\eta}_1} A_{i_1} \cdots E_{z^\#_{\alpha(2k-1)}} \Big) A_{2k-1} E_{\eta_1} x \Big\| \\
&\leqslant 2\varepsilon + \Big\| \sum_{\eta_1 \in I_\varepsilon} \Big(\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots A_{i_1-1} (\eta_1 U)^{n_1} A_{i_1} \cdots U^{n_{\alpha(2k-1)}}
\end{aligned}$$

[1] We reduce the matter to this case as the partitions for which $\alpha(2k-1) = \alpha(2k)$ can be treated by taking into account that the product is jointly continuous in the strong operator topology when restricted to bounded parts.

$$
\begin{aligned}
&- \sum_{z_2,\dots,z_k\in\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)} E_{z^{\#}_{\alpha(1)}}A_1E_{z^{\#}_{\alpha(2)}}\cdots A_{i_1-1}E_{\bar\eta_1}A_{i_1}\cdots E_{z^{\#}_{\alpha(2k-1)}}\Bigg)\Bigg\|\\
&\times\Bigg\|\Bigg(A_{2k-1}E_{\eta_1}x-\sum_{\eta_2\in I_\varepsilon(\eta_1)}E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg)\Bigg\|\\
&+\Bigg\|\sum_{\eta_1\in I_\varepsilon}\sum_{\eta_2\in I_\varepsilon(\eta_1)}\Bigg(\frac{1}{N^k}\sum_{n_1,\dots,n_k=0}^{N-1}U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots A_{i_1-1}(\eta_1U)^{n_1}A_{i_1}\cdots U^{n_{\alpha(2k-1)}}\\
&-\sum_{z_2,\dots,z_k\in\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)}E_{z^{\#}_{\alpha(1)}}A_1E_{z^{\#}_{\alpha(2)}}\cdots A_{i_1-1}E_{\bar\eta_1}A_{i_1}\cdots A_{2k-2}\Bigg)E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg\|\\
&\qquad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\\
&\leqslant 2(2k-i_m)\varepsilon+\Bigg\|\sum_{\eta_1\in I_\varepsilon}\sum_{\eta_2\in I_\varepsilon(\eta_1)}\cdots\sum_{\eta_h\in I_\varepsilon(\eta_1,\eta_2,\dots,\eta_{h-1})}\frac{1}{N^{k-1}}\\
&\times\sum_{n_1,\dots,n_{m-1},n_{m+1},\dots n_k=0}^{N-1}U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots A_{i_m-1}\Bigg(\frac{1}{N}\sum_{n=0}^{N-1}(\eta_mU)^n-E_{\bar\eta_m}\Bigg)\\
&\times A_{i_m}E_{\eta_h}\cdots E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg\|+\Bigg\|\sum_{\eta_1\in I_\varepsilon}\sum_{\eta_2\in I_\varepsilon(\eta_1)}\cdots\sum_{\eta_h\in I_\varepsilon(\eta_1,\eta_2,\dots,\eta_{h-1})}\\
&\times\Bigg(\frac{1}{N^{k-1}}\sum_{n_1,\dots,n_{m-1},n_{m+1},\dots n_k=0}^{N-1}U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(i_m-1)}}\\
&-\sum_{z_{h+1},\dots,z_k\in\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)}E_{z^{\#}_{\alpha(1)}}A_1E_{z^{\#}_{\alpha(2)}}\cdots E_{z^{\#}_{\alpha(i_m-1)}}\Bigg)A_{i_m-1}E_{\bar\eta_1}A_{i_m}\cdots E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg\|\\
&\leqslant[2(2k-i_m)+1]\varepsilon+\Bigg\|\sum_{\eta_1\in I_\varepsilon}\sum_{\eta_2\in I_\varepsilon(\eta_1)}\cdots\sum_{\eta_h\in I_\varepsilon(\eta_1,\eta_2,\dots,\eta_{h-1})}\\
&\times\Bigg(\frac{1}{N^{k-1}}\sum_{n_1,\dots,n_{m-1},n_{m+1},\dots n_k=0}^{N-1}U^{n_{\alpha(1)}}A_1U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(i_m-1)}}\\
&-\sum_{z_{h+1},\dots,z_k\in\sigma^{\mathrm{a}}_{\mathrm{pp}}(U)}E_{z^{\#}_{\alpha(1)}}A_1E_{z^{\#}_{\alpha(2)}}\cdots E_{z^{\#}_{\alpha(i_m-1)}}\Bigg)A_{i_m-1}E_{\bar\eta_1}A_{i_m}\cdots E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg\|\\
&\qquad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\\
&\leqslant(3k-1)\varepsilon+\Bigg\|\sum_{\eta_1\in I_\varepsilon}\sum_{\eta_2\in I_\varepsilon(\eta_1)}\cdots\sum_{\eta_k\in I_\varepsilon(\eta_1,\eta_2,\dots,\eta_{k-1})}\Bigg(\frac{1}{N}\sum_{n=0}^{N-1}(\eta_kU)^n-E_{\bar\eta_k}\Bigg)\\
&\times A_1E_{\eta^{\#}_{\alpha(2)}}\cdots E_{\eta_2}A_{2k-1}E_{\eta_1}x\Bigg\|\leqslant 3k\varepsilon.\qquad\square
\end{aligned}
$$

4. Multiple correlations

The study of multiple correlations is a standard matter of interest in classical and quantum ergodic theory for several application to various fields. For example they are of interest to investigate the chaotic behavior of dynamical systems. We also mention the natural applications to quantum statistical mechanics, number theory, probability. The reader is referred to [5,7] for further details (see also [3,4] for some partial results involving multiple correlations and recurrence). The present analysis allows us to study the limit of the Cesaro mean of several multiple correlations.

We start with a C^*-dynamical system $(\mathfrak{A}, \gamma, \omega)$ made of a C^*-algebra $\mathfrak{A}$, an automorphism γ of $\mathfrak{A}$, and finally a state ω on $\mathfrak{A}$ which is invariant under γ. Consider the covariant GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega, U)$ (cf. [8]) associated to the C^*-dynamical system under consideration.

Theorem 4.1. *Under the above notations, suppose that U implementing γ on $\mathcal{H}_\omega$ is almost periodic. Then for each pair partition $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$, we have*

$$\lim_N \frac{1}{N^k} \sum_{n_1,\ldots,n_k=0}^{N-1} \omega\left(A_0 \gamma^{n_{\alpha(1)}}(A_1)\gamma^{n_{\alpha(1)}+n_{\alpha(2)}}(A_2)\cdots\gamma^{\left(\sum_{l=1}^{2k-1} n_{\alpha(l)}\right)}(A_{2k-1})\gamma^{\left(\sum_{i=1}^{k} 2n_i\right)}(A_{2k})\right)$$
$$= \left\langle \pi_\omega(A_0) S_{\alpha;\pi_\omega(A_1),\ldots,\pi_\omega(A_{2k-1})} \pi_\omega(A_{2k})\Omega, \Omega\right\rangle.$$

Proof. The proof directly follows from Theorem 3.1, by taking into account that

$$\omega\left(A_0 \gamma^{n_{\alpha(1)}}(A_1)\gamma^{n_{\alpha(1)}+n_{\alpha(2)}}(A_2)\cdots\gamma^{(\sum_{l=1}^{2k-1} n_{\alpha(l)})}(A_{2k-1})\gamma^{(\sum_{i=1}^{k} 2n_i)}(A_{2k})\right)$$
$$= \langle \pi_\omega(A_0)U^{n_{\alpha(1)}}\pi_\omega(A_1)U^{n_{\alpha(2)}}\cdots U^{n_{\alpha(2k-1)}}\pi_\omega(A_{2k-1})U^{n_{\alpha(2k)}}\pi_\omega(A_{2k})\Omega, \Omega\rangle. \qquad \square$$

For the sake of completeness, we notice that the same result holds true for C^*-dynamical systems of compact operators, without any condition on the unitary U implementing the dynamics. Namely, let $(\mathcal{K}(\mathcal{H}), \gamma)$ be a C^*-dynamical system based on the algebra of all the compact operators acting on the Hilbert space $\mathcal{H}$. It is well known that $\gamma = \mathrm{ad}(U)$, that is it is unitarily implemented on $\mathcal{H}$ by the adjoint action of some unitary U, uniquely determined up to a phase. We denote by γ the adjoint action $\mathrm{ad}(U) \equiv \gamma^{**}$ on $\mathcal{B}(\mathcal{H})$ as well.[2]

Let Tr be the canonical trace on $\mathcal{B}(\mathcal{H})$, and consider a positive normalized trace class operator T acting on $\mathcal{H}$ such that $s(T) \leqslant E_1$, $s(T)$ and E_1 being the support of T and the spectral projection onto the invariant vectors for U, respectively. It is easy to show that

$$UT = T = TU. \tag{4.1}$$

Let $\omega_T \in \mathcal{B}(\mathcal{H})_*$ be the state defined as

$$\omega_T(A) := \mathrm{Tr}(TA), \quad A \in \mathcal{B}(\mathcal{H}).$$

Thanks to (4.1), it is invariant under γ, as its restriction to $\mathcal{K}(\mathcal{H})$ denoted with an abuse of notation also by ω_T. The following result parallels Theorem 4.1.

Theorem 4.2. *For each pair partition $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$, $\{A_0, A_{2k}\} \subset \mathcal{B}(\mathcal{H})$ and $\{A_1, \ldots A_{2k-1}\} \subset \mathcal{K}(\mathcal{H})$, we have*

[2] It is enough to consider the double transpose $\gamma^{**} \in \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$, which is an automorphism of $\mathcal{B}(\mathcal{H})$, and therefore inner. The previous mentioned phase factor is inessential for our computations.

$$
\begin{aligned}
&\lim_N \frac{1}{N^k} \sum_{n_1,\dots,n_k=0}^{N-1} \omega_T \Big(A_0 \gamma^{n_{\alpha(1)}}(A_1)\gamma^{n_{\alpha(1)}+n_{\alpha(2)}}(A_2) \times \cdots \\
&\times \gamma^{\left(\sum_{l=1}^{2k-1} n_{\alpha(l)}\right)}(A_{2k-1})\gamma^{\left(\sum_{i=1}^{k} 2n_i\right)}(A_{2k})\Big) \\
&= \omega_T \Big(A_0 S_{\alpha;A_1,\dots,A_{2k-1}} A_{2k}\Big).
\end{aligned} \tag{4.2}
$$

Proof. By (4.1), we get

$$
\begin{aligned}
&\omega_T \Big(A_0 \gamma^{n_{\alpha(1)}}(A_1)\gamma^{n_{\alpha(1)}+n_{\alpha(2)}}(A_2) \cdots \gamma^{\left(\sum_{l=1}^{2k-1} n_{\alpha(l)}\right)}(A_{2k-1})\gamma^{\left(\sum_{i=1}^{k} 2n_i\right)}(A_{2k})\Big) \\
&= \omega_T \left(A_0 U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(2k-1)}} A_{2k-1} U^{n_{\alpha(2k)}} A_{2k}\right).
\end{aligned}
$$

The proof directly follows from Theorem 2.6 of [2] by approximating the trace class operator T by a finite rank one. $\square$

We conclude by noticing that Theorem 2.6 of [2] allows us to treat all the multiple correlations arising from any general partition of any set of m points in k parts, for dynamical systems based on the compact operators. In the case of non pair partitions, it is not immediate to provide a general formula for the limit in (4.2).

5. Appendix

Unfortunately, a proof of our main theorem (or of the estimation in Lemma 2.1) based on the induction principle works well only for non crossing partitions.[3] Due to entanglement and to the fact that the mean ergodic theorem (1.2) holds true only in the strong operator topology, any attempt to provide any kind of induction proof of Theorem 3.1 produces essentially the same complexity as the proof presented in this paper. While keeping the original proof, to show how the last is working, we report the particular case of the entangled partition $\alpha = \{1, 2, 1, 3, 2, 3\}$.

Fix $\varepsilon > 0$, and suppose that $A, B, C, D, F \in \mathcal{B}(\mathcal{H})$ have norm one. Let I_ε be such that

$$
\left\| x - \sum_{\sigma \in I_\varepsilon} E_\sigma x \right\| < \varepsilon.
$$

For each $\sigma \in I_\varepsilon$, let $I_\varepsilon(\sigma)$ be such that

$$
\left\| FE_\sigma x - \sum_{\tau \in I_\varepsilon(\sigma)} E_\tau FE_\sigma x \right\| < \frac{\varepsilon}{|I_\varepsilon|}.
$$

Finally, for each $\sigma \in I_\varepsilon$, $\tau \in I_\varepsilon(\sigma)$, let $I_\varepsilon(\sigma, \tau)$ be such that

$$
\left\| CE_{\bar{\sigma}} DE_\tau FE_\sigma x - \sum_{\rho \in I_\varepsilon(\sigma,\tau)} E_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|}.
$$

In addition, by the mean ergodic theorem (1.2), choose N_ε such that

$$
\left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} (\sigma U)^n - E_{\bar{\sigma}} \right) DE_\tau FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|},
$$

[3] Among the set of pair partitions of four elements, $\{1, 2, 2, 1\}$ and $\{1, 1, 2, 2\}$ are non-crossing, whereas the remaining one $\{1, 2, 1, 2\}$ is crossing. The reader is referred to [1] for the abstract definition of crossing partitions.

$$\left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} (\tau U)^n - E_{\bar{\tau}} \right) BE_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} |I_\varepsilon(\sigma, \tau)|},$$

$$\left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} (\rho U)^n - E_{\bar{\rho}} \right) AE_{\bar{\tau}} BE_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} |I_\varepsilon(\sigma, \tau)|},$$

whenever $N > N_\varepsilon$, $\sigma \in I_\varepsilon$, $\tau \in I_\varepsilon(\sigma)$, and $\rho \in I_\varepsilon(\sigma, \tau)$. Let $\beta = \{1, 2, 1, 2\}$, and $\gamma = \{1, 1\}$. We then obtain for each $N > N_\varepsilon$,

$$\begin{aligned}
&\left\| \frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k AU^m BU^k CU^n DU^m FU^n x - S_{\alpha;A,B,C,D,F} x \right\| \\
&\leqslant \left\| \frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k AU^m BU^k CU^n DU^m FU^n - S_{\alpha;A,B,C,D,F} \right\| \left\| x - \sum_{\sigma \in I_\varepsilon} E_\sigma x \right\| \\
&\quad + \left\| \sum_{\sigma \in I_\varepsilon} \left(\frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k AU^m BU^k C(\sigma U)^n DU^m - S_{\beta;A,B,CE_{\bar{\sigma}} D} \right) FE_\sigma x \right\| \\
&\leqslant 2\varepsilon + \left\| \frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k AU^m BU^k C(\sigma U)^n DU^m - S_{\beta;A,B,CE_{\bar{\sigma}} D} \right\| \\
&\quad \times \textstyle\sum_{\sigma \in I_\varepsilon} \left\| FE_\sigma x - \displaystyle\sum_{\tau \in I_\varepsilon(\sigma)} E_\tau FE_\sigma x \right\| \\
&\quad + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \left(\frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k A(\tau U)^m BU^k C(\sigma U)^n - S_{\gamma;AE_{\bar{\tau}} B} CE_{\bar{\sigma}} \right) DE_\tau FE_\sigma x \right\| \\
&\leqslant 4\varepsilon + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \left[\frac{1}{N^2} \sum_{k,m=0}^{N-1} U^k A(\tau U)^m BU^k C \left(\frac{1}{N} \sum_{n=0}^{N-1} (\sigma U)^n - E_{\bar{\sigma}} \right) \right] DE_\tau FE_\sigma x \right\| \\
&\quad + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \left(\frac{1}{N^2} \sum_{k,m=0}^{N-1} U^k A(\tau U)^m BU^k - S_{\gamma;AE_{\bar{\tau}} B} \right) CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| \\
&\leqslant 5\varepsilon + \left\| \frac{1}{N^2} \sum_{k,m=0}^{N-1} U^k A(\tau U)^m BU^k - S_{\gamma;AE_{\bar{\tau}} B} \right\| \\
&\quad \times \left\| CE_{\bar{\sigma}} DE_\tau FE_\sigma x - \sum_{\rho \in I_\varepsilon(\sigma,\tau)} E_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| \\
&\quad + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \sum_{\rho \in I_\varepsilon(\sigma,\tau)} \left(\frac{1}{N^2} \sum_{k,m=0}^{N-1} (\rho U)^k A(\tau U)^m - E_{\bar{\rho}} AE_{\bar{\tau}} \right) BE_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| \\
&\leqslant 7\varepsilon + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \sum_{\rho \in I_\varepsilon(\sigma,\tau)} \left[\frac{1}{N} \sum_{k=0}^{N-1} (\rho U)^k A \left(\frac{1}{N} \sum_{m=0}^{N-1} (\tau U)^m - E_{\bar{\tau}} \right) \right] BE_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| \\
&\quad + \left\| \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \sum_{\rho \in I_\varepsilon(\sigma,\tau)} \left(\frac{1}{N} \sum_{n=0}^{N-1} (\rho U)^n - E_{\bar{\rho}} \right) AE_{\bar{\tau}} BE_\rho CE_{\bar{\sigma}} DE_\tau FE_\sigma x \right\| \leqslant 9\varepsilon.
\end{aligned}$$

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